Unobservable investments, trade efficiency and search frictions

Yujing Xu

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May 2021

Abstract

Information structure can affect market efficiency by influencing participants' investment incentives. In this case, the entrants' type distribution and the market trade efficiency are both affected by the information structure, so they are jointly determined and correlated. In order to investigate the effects of asymmetric information, this paper uses a random search model of ex ante investments and trade efficiency, assuming that the amount of investment remains the investor's private information. We show that the ex-ante payoffs are equivalent to the equilibrium payoffs when investments are observable. In the unique steady state equilibrium, non-degenerate investment distribution and price distribution emerge simultaneously with ex ante identical agents. The investments motivated by unobservability fail to improve social welfare due to the mismatches caused by unobservability. Allocating positive bargaining power to investors alleviates the mismatch problem and improves social welfare.

Keywords: Search Frictions, Unobservable Investments, Efficiency

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*Shenzhen Audencia Business School & WeBank Institute of Fintech, Shenzhen University. E-mail: yujingxu@szu.edu.cn.
The author is indebted to Hugo Hopenhayn, Simon Board, Moritz Meyer-ter-Vehn and Ichiro Obara for their guidance. The author is also grateful to Pierre-Olivier Weill, Joseph Ostroy, Bill Zame, Marek Pycia, Michael Peters, Hao Li, Kim Sau Chung, Wing Suen, Kenneth Mirkin, Cheng Chen and participants of various seminars and conferences. This work was supported by UCLA Dissertation Year Fellowship and the General Research Fund of HKRGC #17500620.
(**JEL classification: D83; L11**)

1 Introduction

Participants in large markets are often heterogenous. With a large number of participants in such a market, a participant would know better about her own characteristics than a random trading partner would. For instance, freelancers (or independent contractors) have different costs for delivering certain tasks, which is their private information. Whether participants possess private information or not has been proven to significantly affect market outcomes in the literature on Dynamic Matching and Bargaining Games (DMBG) without complementarities. For example, Lauermann (2013) shows that trading outcomes become competitive as search frictions vanish whenever participants' preferences are their private information, and that such a convergence would likely fail when preferences are observable. This result Holds in a very general class of matching-and-bargaining games, and summarizes the importance of asymmetric information for efficiency in such a decentralized market.

One implicit assumption in most of the literature, though, is that the entrants' type distribution is exogenous and held fixed when comparing trading outcomes with different information structures. In reality, entrants might invest in order to change their characteristics.[1] These investments are motivated by future gains that depend crucially on the information structure. In this case, asymmetric information could affect market outcomes through this additional channel. The overall impacts of asymmetric information need to be re-examined.[2]

In order to address this issue, we investigate a random search model with ex ante identical but ex post heterogeneous agents in which the heterogeneity arises endogenously from agents' ex ante investments. We explore the effects of the correlation between type distribution and trade outcomes, and arrive at a drastically different prediction for how asymmetric information affects market outcomes than has been seen before in the existing literature. In this paper, we show

[1]For example, freelancers acquire costly skills or tools (such as softwares) to lower the cost of delivering outputs, according to the survey conducted by Upwork and the Freelancers Union (https://www.slideshare.net/upwork/freelancing-in-america-2018-120288770/1).

[2]The idea that whether participants' type distribution is endogenous or exogenous could greatly affect market outcomes has also been noticed in earlier research. In a labor search model, Acemoglu and Shimer (1999) show that while Hosios' condition guarantees efficiency when firms' productivity is exogenous, it is not sufficient for efficiency when firms can invest to enhance productivity. Additional discussion can be found in the "Related Literature" section.

that the ex ante payoffs are equivalent to those with observable investments, given any level of search frictions, and that the effect of asymmetric information on trade outcomes and on investment incentives cancel each other out.

The infinite-horizon random search model used in this paper is a typical matching-and-bargaining game except that market participants can invest before entering. New buyer and seller entrants in each period are identical ex ante. A buyer entrant demands one unit of output and receives utility $y_0 > 0$ from consumption. A seller entrant is endowed with the technology to produce one unit of output at cost $x_0 \in (0, y_0)$ and can lower the production cost by investing. After the investments have been sunk, all incumbents randomly form one-buyer-to-one-seller pairs. Within each pair, the buyer makes a take-it-or-leave-it offer without observing the seller's investments. If the offer is accepted, production takes place and both leave the market permanently. Otherwise, the pair is dissolved and both search in the next period. We assume that agents are impatient and that the time between two periods is the source of the search friction. In the model, agents are referred to as buyers and sellers, but they may also be consumers and retailers, clients and freelancers,[3] and so on.

A key assumption in the model is that the investment is unobservable. We first show that agents' ex ante payoffs are equivalent to those when the investment is observable, in which case no sellers invest. In other words, private information neither benefits the investors nor improves social welfare. Buyers can correctly anticipate the lowest amount invested in equilibrium and hence are able to fully extract the least efficient sellers when they make offers as if the investments were observable. The sellers' lowest amount of investment is therefore 0. The sellers' ex ante payoff, and, according to their optimal investment decision, any seller's ex ante payoff, is equivalent to that with observable investments. Buyers who propose the reserve price of the least efficient sellers offer the same price and trade immediately, as in the case of observable investments. The buyers' payoff, and according to their optimal pricing decision, any buyer's payoff, is equivalent to that

[3]Our model best applies to the freelance economy when looking at jobs with straightforward goals, such as typing, coding and data collecting. Other types of tasks, for example where a freelancer's investments also benefit clients by improving the quality of the output, are beyond the scope of this paper.

with observable investments.

The ex ante payoff equivalence property stems from two opposing effects caused by unobservability. The *extraction-limitation effect* incentivizes sellers to invest positive amounts. Because the investments are unobservable, a buyer who wants to extract more surplus from the paired seller's investment risks losing the trading opportunity when offering a lower price. This trade-off limits the buyers' ability to extract surplus. In equilibrium, the distribution of the production costs is such that buyers are indifferent between offering higher prices in order to secure trade and offering lower prices in order to extract more surplus. As a result, sellers could receive information rent when they receive high offers, which incentivizes them to invest. The investments generate ex post gains when trade takes place. In equilibrium, unobservability gives rise to a non-degenerate distribution of production costs and a non-degenerate distribution of price offers with ex ante identical buyers and sellers. This means that the price offer and the reserve price in a pair may be mismatched, resulting in no trade, despite the fact that the surplus from trading immediately is strictly positive. The *mismatch effect* caused by unobservability leads to a delay in trade and a delay in the realization of ex post gains from investments, which, in turn, completely dissipates any gain from investments motivated by private information.

The ex ante payoff of sellers is 0 and that of buyers is $y_0 - x_0$, which are constant given any level of search frictions. This implies that social welfare is not constraint efficient even when the search friction vanishes. This contradicts the robust prediction in the literature on DMBG without complementarity that the equilibrium converges to the competitive limit as search frictions vanish. In order to solve this puzzle, we investigate how a reduction in the search friction might affect trade efficiencies and investment incentives. We find that this would lead to a higher degree of mismatch: buyers price more aggressively and the market accumulates more high-cost sellers. Contrary to our result, trade becomes efficient as search frictions vanish in DMBG. This explains the different predictions in terms of social welfare. Moreover, we also find that investments become socially efficient as the search friction vanishes, although it does not contribute to social welfare at all. This is due to selection in a large market: seller incumbents who have invested more are less likely to be mismatched, so they leave the market faster. This selection is strongest as the search friction vanishes, which means that almost all of the seller entrants must invest the socially efficient amount to preserve the steady state. This observation suggests that when evaluating the efficiency of a market empirically, it is important to make use of both the data on the entrants' type distribution and the data on trade efficiency. Otherwise, one may wrongly conclude that a market with almost efficient seller entrants like this one must always generate high social welfare.

We then extend the basic model to include random proposers.[4] We learned three things about allocating some bargaining power to sellers. First, the lowest amount of investment in the mixed strategy will become strictly positive because, when sellers propose, they can claim the residual gains from investment. Second, the ex ante payoff equivalence property continues to hold and so it is not restricted to the knife-edge case of one-sided proposing. Third, when sellers have a positive probability of proposing in one period, the lowest investment amounts and the social welfare increase as the search friction is reduced. They converge to the first best level pointwise as the friction vanishes. In the appendix, we also explore how the insights from the basic model help us to understand similar market settings, such as markets with two-sided investments and randomly observable investments.

Related Literature

This paper is closely related to the literature on DMBG without complementarities, as discussed earlier. Private information plays a key role in determining the trading outcomes, as highlighted in Satterthwaite and Shneyerov (2007), Shneyerov and Wong (2010a) and Lauermann (2012) (2013), among others. With the additional investment stage in this paper, both the entrants' type distribution and the trading outcomes are affected by private information. This essentially introduces a correlation between the two, which is absent in the earlier literature. We show that this correlation could lead to new conclusions about the net effect of private information as well as market efficiency as search frictions vanish.

This paper explores investment incentives, and so it is related to the literature on the hold-up

[4]For example, sometimes it is the freelancer who proposes the terms of the contract instead of the client.

problem (Grout (1984), Grossman and Hart (1986), Hart and Moore (1990), Hermalin and Katz (2009), Hermalin (2013) etc.). Models of the hold-up problem largely focus on one-to-one trade with relationship-specific investments, while in our model, agents participate in a large market with general investments. Despite this, we show that the sellers who invest the lowest amount are still held up. In this thread of the literature, the most related paper is Gul (2001). He looks at a Coasian setting (i.e., the seller makes one-sided repeated offers) where a buyer's valuation is determined by her investments prior to the bargaining process, which is assumed to be unobservable to the seller. We share Gul's conclusion that the equilibrium strategies feature double-mixing. Moreover, the payoff equivalence property also holds in Gul's paper when the seller only proposes once. This one-round proposing is essentially the same as our basic model assuming $\beta = 0$. On the other hand, the predictions on how the equilibrium changes as the environment becomes more competitive[5] differ due to different market settings (one-to-one relation v.s. a large market).

Firstly, although the investment strategy approaches efficiency in both settings, the mechanisms behind the convergence are not the same. The investment incentives in Gul (2001) result from the asymmetric information and the Coasian effect. In our basic model, the convergence is due to the extreme selection of the market in the steady state as the search friction vanishes. Secondly, there are more mismatches in our model as the search friction diminishes because the market accumulates more high cost sellers due to selection in the market. This selection is absent in Gul (2001) without repeated entry. Then, with the almost efficient buyer in Gul's paper, there is almost no delay in bargaining. Thirdly, because of the difference in the efficiency of trade, the efficient investments motivated by information rent generate the first best welfare in Gul (2001) while they do not contribute to the welfare at all in our basic model. In other words, our results differ from Gul (2001) in that, in environments captured by our basic model, private information may not be able to restore efficiency.

We are not the first paper to investigate investment incentives and trading outcomes jointly in a dynamic environment with search frictions. A non-exhaustive list includes Masters (1998, 2011),

[5] Being more competitive means that the time between two rounds shrinks to zero (in Gul [2001]) and that the time between two periods shrinks to zero (in our model).

Smith (1999), Acemoglu (1996), Acemoglu and Shimer (1999), Charlot, Decreuse and Granier (2005), Charlot and Decreuse (2010), Davis (2001), Kurmann (2014), Flinn and Mullins (2015) and Jerez (2017). What we share in common is that, when participants' types are determined by ex ante investment rather than exogenously given, efficiency cannot be reached under usual conditions. On the other hand, none of the above mentioned literature looks at agents with private information. What we add to this literature is how asymmetric information affects investment incentives, trade efficiency, and welfare.

This paper is also related to random search models with heterogeneous agents. Albrecht and Vroman (1992) demonstrate that, when seller entrants are exogenously heterogeneous, any steady state equilibrium must feature a non-degenerate price distribution. Our paper complements theirs by showing that the sellers' heterogeneity can emerge endogenously, precisely because buyers offer diverse prices in equilibrium.

The rest of the paper is organized as follows. The model is introduced in Section 2, in which we also characterize two benchmark specifications: that of the first best and that of observable investments. Section 3 sets out the equilibrium conditions, derives the ex ante payoff equivalence property, and proves the existence and uniqueness of the steady state equilibrium. The extraction-limitation effect and the mismatch effect are explained in Section 4 while the impacts of reduced search frictions are explored in Section 5. Section 6 considers an extension about random proposers. Section 7 concludes the paper.

2 The Model and Benchmark Specifications

2.1 The Model

We consider a discrete-time, infinite-horizon random search model with ex ante investments and exogenous entry. Throughout this paper, we will focus on the steady state equilibrium and omit the time index.

At the beginning of each period, a unit measure of buyers and a unit measure of sellers enter the market. Each buyer demands one unit of the product and enjoys utility $y_0 > 0$ from consumption. Each seller entrant is endowed with the technology that produces one unit of output at a cost $x_0 > 0$. We focus on the "gap" case throughout this paper, i.e., when the (minimum) surplus from trade $y_0 - x_0$ is strictly positive.[6]

Before entering the market, a seller can invest $c(x)$ to lower the production cost to $x \geq 0$ once and for all, where the function $c : [0, x_0] \to \mathbb{R}$ is strictly decreasing, continuously differentiable, strictly convex with $c(x_0) = c'(x_0) = 0$, and $c'(0) < -1$.[7] We will call this seller a type x seller, although all sellers start out identical. Entrants then join the agents who stayed from the last period and together they form the incumbents of the current period. All incumbents form one-buyer-to-one-seller pairs randomly.[8] The buyer in each pair makes a take-it-or-leave-it offer p and the seller decides whether to accept it. We assume that buyers have no information about investments when making price offers, which is crucial to the results. Buyers only know the probability that an incumbent seller has a type weakly smaller than x, which is denoted as $F(x)$. We also call F the CDF of the stationary cost distribution. In addition, the matching is anonymous.

If the offer is accepted, then one unit of output is produced and sold, which leaves the seller payoff $p - x$ and the buyer payoff $y_0 - p$. Both agents exit the market permanently. Otherwise, the pair is dissolved and both agents will search in the next period. The time between two successive periods is t. We assume that all agents are impatient and share the same discount rate r. The discount factor is $\beta = e^{-rt} \in [0, 1)$. We refer to the search friction as small if t is small or, equivalently, if β is large.

A seller's strategy consists, first, of an investment strategy governed by a CDF F_e, where $F_e(x)$

[6] In the online appendix, we show that the market is inactive in equilibrium with the "no-gap" case, i.e., when $y_0 - x_0 \leq 0$.

[7] Assume that $c'(x_0) = 0$ and $c'(0) < -1$ are simplifications without loss of generality. As will become clear later, the assumption $c'(x_0) = 0$ ensures that zero measure of sellers will make no investment, and the assumption $c'(0) < -1$ implies that the socially optimal investment amount (defined later) is interior. The main conclusions of this paper, including the ex ante payoff equivalence property, are robust to alternative assumptions.

[8] With bilateral trading and equal measures of buyer- and seller-entrants, the two sides of the market are of equal size. Then all incumbents forming one-buyer-to-one-seller pairs is feasible. The alternative assumption of a probabilistic matching function, as is often adopted in the literature, would not change the results qualitatively.

equals the probability that he[9] invests weakly more than $c(x)$.[10] Secondly, it consists of a reserve price function r_S, where $r_S(x)$ equals the lowest price that a type x seller is willing to accept. A buyer decides what price to offer, which is governed by a function H, where $H(p)$ equals the probability of offering a price strictly lower than p.

2.2 Two Benchmark Specifications

Benchmark Case 1: the Planner's Solution

We first characterize the efficient allocation, which consists of both efficient investment and efficient trading. At the search stage of each period, a social planner would find it optimal to always conduct trade given any cost distribution, as the surplus from trade is always positive and postponing trading is costly due to discounting. Given that trades take place immediately, investing $c(x)$ increases the social surplus by $x_0 - x$. A social planner therefore chooses $x^* \in (0, x_0)$ implicitly defined by $c'(x^*) = -1$ to equate the marginal cost with the marginal benefit of the investments. The social welfare generated from the planner's solution is denoted as s^* and $s^* = y_0 - x^* - c(x^*)$.

Benchmark Case 2: Observable Investments

Assume that investments are observable to buyers. By comparing the basic model with this benchmark case, we would like to identify the effects of private information on the investment incentives, trade efficiencies, payoffs and social welfare.

The buyers have all of the bargaining power. Following the same logic as in Diamond (1971), a seller receives a search stage payoff of zero regardless of his production cost. Because buyers can observe the seller's production cost, the paired buyer and buyers in all future matches offer exactly the seller's production cost plus the discounted continuation payoff, which drives the continuation payoff down to zero with infinitely repeated discounting and β being smaller than 1. Therefore, the equilibrium is unique with no sellers investing, all buyers offering price x_0, and no delay in trade.

[9] We refer to a seller as "he" and a buyer as "she".

[10] Here, we put no restriction on F_e. For example, the support of F_e can be degenerate. However, as will become clear later, the investment strategy indeed has a non-degenerate support. The same applies to H as defined later.

The holdup problem remains, although the investment is not relationship-specific in this large market. We will contrast this equilibrium of no investment and no delay with the equilibrium of positive investment and positive delay in the basic model when investments become unobservable.

In this benchmark case, a seller's ex ante payoff, denoted as v, is zero, and a buyer's ex ante payoff, denoted as π, is $y_0 - x_0$. Social welfare is defined as the sum of the buyer's and the seller's ex ante payoff, which equals $y_0 - x_0 \equiv s_0$.

3 The Steady State Equilibrium

Let us now solve for the steady state equilibrium in the decentralized market.

3.1 Equilibrium Conditions

The Seller Incumbent's Problem A type x seller-incumbent chooses the reserve price $r_S(x)$ to maximize the search stage payoff $U(x)$. The domain of the function U and r_S is the same as the support of F. Given the price offer function H, the seller's trading probability is $1 - H(r_S(x))$, which is decreasing in $r_S(x)$. The maximization problem of a type x seller is:

$$U(x) = \max_r \{(E(p \mid p \geq r) - x)(1 - H(r)) + H(r)\beta U(x)\}. \tag{1}$$

Solving the above problem, the reserve price $r_S(x)$ should exactly cover the opportunity cost of trading, i.e., the production cost x plus the discounted continuation payoff:

$$r_S(x) = x + \beta U(x). \tag{2}$$

Lemma 1. *In any steady state equilibrium,*

1. *the payoff function U is strictly decreasing and continuous in x and the reserve price function r_S is strictly increasing and continuous in x;*

2. *the payoff function U is linear over an interval $[x^1, x^2]$ if no buyer offers any price strictly between $r_S(x^1)$ and $r_S(x^2)$, i.e., $H(r_S(x^2)) - H(r_S(x^1)) - Pr(p = r_S(x^1)) = 0$.*

Unless otherwise mentioned, all of the proofs are gathered in Appendix B. A more efficient seller should have a higher search stage payoff because his production cost is lower, and should have a lower reserve price because the opportunity cost for trading is lower. We also see from the second part of the lemma that the shape of U depends on H. The direct benefit of the investments on information rent is linear because the production cost is reduced linearly. Therefore, U can be strictly convex over an interval $[x^1, x^2]$ only if the type x^1 seller also benefits indirectly from his higher investments through the increased probability of receiving information rent. This probability is higher if the type x^1 seller is more likely to get an offer strictly that is higher than his reserve price, which amounts to some buyers proposing prices strictly between $r_S(x^1)$ and $r_S(x^2)$.

Because sellers are identical ex ante, any x on the support of F_e should yield the same ex ante payoff v, which is computed as $U(x) - c(x)$ for x. That is, the following indifference condition should hold:

$$U(x) - c(x) = v \text{ for any } x \text{ on the support of } F_e \text{, and} \tag{3}$$
$$U(x) - c(x) \leq v \text{ for any } x \text{ not on the support of } F_e.$$

The Buyer's Problem A buyer chooses what price to offer. Define function $\hat{x}$ as the inverse function of r_S, where a seller with a production cost higher than $\hat{x}(p)$ would reject price p. Lemma 1 implies that $\hat{x}$ is well defined, continuous and strictly increasing over the relevant range. For a given F, a buyer offering p trades with probability $F(\hat{x}(p))$ in each period. Then the equilibrium payoff of a buyer equals

$$\pi = \max_p \{(y_0 - p)F(\hat{x}(p)) + (1 - F(\hat{x}(p)))\beta\pi\}. \tag{4}$$

In equilibrium, F must be such that it makes a buyer indifferent to any p on the support of H.

In other words,

$$(y_0 - p)F(\hat{x}(p)) + (1 - F(\hat{x}(p)))\beta\pi = \pi \text{ for any } p \text{ on the support of } H\text{, and} \tag{5}$$
$$(y_0 - p)F(\hat{x}(p)) + (1 - F(\hat{x}(p)))\beta\pi \leq \pi \text{ for any } p \text{ not on the support of } H.$$

The Seller's Investment Strategy The last piece of the equations is the distribution of the seller entrants' production cost, which is also the sellers' investment strategy.[11] In a steady state equilibrium, the measure of outflow of any type must equal the measure of inflow of the same type to preserve the stationary distribution over time. A type x seller leaves the market if he receives an offer weakly higher than $r_S(x)$, which happens with probability $1 - H(r_S(x))$. Meanwhile, the measure of entrants with a cost lower than x is $F_e(x)$. Denote the lowest production cost on the support as $\underline{x}$ and the highest as $\bar{x}$. The steady state equilibrium requires that, for any x on the support,

$$F_e(x) = \frac{F(x) - \int_{\underline{x}}^{x} H(r_S(\tilde{x}))dF(\tilde{x})}{1 - \int_{\underline{x}}^{\bar{x}} H(r_S(\tilde{x}))dF(\tilde{x})}. \tag{6}$$

We summarize the above description with a definition of the steady state equilibrium.

Definition 1. *A steady state equilibrium consists of* $\{U, v, \pi, r_S, \hat{x}, F_e, F, H\}$*, such that*

1. *function* U *is defined in (1) given* H *and* v *is computed in (3);*
2. *function* r_S *is defined in (2) and* $\hat{x}(p) = r_S^{-1}(p)$ *for any* p *on the support of* H*;*
3. *buyer's payoff* π *is defined in (4) given* F *and* $\hat{x}$*;*
4. *the indifference conditions (3) and (5) are satisfied; and*
5. *equation (6) holds for any* x *on the support of* F*.*

[11] As there is a continuum of entrants, we obtain the equivalence between the distribution of entrants' costs and the investment strategy when we abuse the law of large numbers as usual.

3.2 Ex Ante Payoff Equivalence

When investments are observable, we have shown that any gains from investments are fully extracted by buyers. Consequently, sellers do not invest and the equilibrium outcomes are inefficient. When investments are unobservable, it is not surprising to find that investments become more efficient as shown in our later analysis, because sellers gain information rent. However, it turns out that the ex ante payoffs of buyers and sellers, as well as the social welfare, are all the same as in the observable benchmark case. This contradicts the prediction in both the DMBG and the holdup literature (such as Lauermann [2013] and Gul [2001]) that different information structures generate different welfare outcomes.

To see why this is the case, notice that, although an individual seller's production cost is unobservable, buyers know, in equilibrium, that the highest production cost is $\bar{x}$. The highest price they propose is therefore $r_S(\bar{x})$. Then a type $\bar{x}$ seller is either fully extracted, earning no information rent, or unable to trade when offered lower prices. Unobservability does not raise the search stage payoff for the least efficient sellers in the market; it continues to be zero. Those sellers would invest zero and the holdup problem would remain.

Lemma 2. *In any steady state equilibrium with unobservable investments,* $U(\bar{x}) = 0$*,* $\bar{x} = x_0$ *and* $r_S(\bar{x}) = x_0$.

The result of this lemma means that the ex ante payoff of all sellers can be computed as $v = U(x_0) - c(x_0) = 0$ based on the indifference condition (3). In other words, though private information gives rise to positive ex post information rent, it does not add to the seller's ex ante payoff. This result depends little on the details of the model except that the investment completely determines the ex post production cost[12] and that the buyers have all the bargaining power. For instance, the argument does not depend on the shape of the investment cost function.[13]

For buyers, those who offer the highest reserve price $r_S(\bar{x})$ trade without delay, and gain payoff $\pi = y_0 - r_S(\bar{x}) = y_0 - x_0$. Based on indifference condition (5), this is the equilibrium payoff of

[12]This assumption is also used in Gul (2001).

[13]Even the standard assumptions imposed on function c earlier are not necessary for this particular result, except that it should be non-negative.

all buyers regardless of the prices they are offering. The payoffs are equivalent in the observable and unobservable cases.

Proposition 1. ***(Ex Ante Payoff Equivalence)*** *In any steady state equilibrium with unobservable investments,* $v = 0$ *and* $s = \pi = s_0$ *for any* $\beta \in [0, 1)$. *These payoffs are equivalent to the ex ante payoffs with observable investments.*

In Section 6, we will show how the ex ante payoff equivalence property extends to cases where sellers have a positive probability of proposing, although their ex ante payoff may be positive. Next, we characterize the equilibrium F, F_e and H in support of this property.

3.3 the Steady State Distributions

Let us first illustrate the dynamics of a steady state equilibrium. In any period, the stationary cost distribution F is such that it keeps buyers indifferent. The seller's reserve price r_S, the buyer's pricing strategy H, and F jointly determine the cost distribution of those who trade and exit. At the beginning of the next period, based on the sellers' indifference condition, the new generation of seller entrants will choose the investment strategy F_e so that they exactly replace those who exited. This way, the stationary cost distribution is preserved over time. Now, we can show the following.

Lemma 3. *In any steady state equilibrium, the price function* H*, the seller's investment strategy* F_e *and the stationary cost distribution* F *have the following properties:*

1. F *and* F_e *have support* $[x^*, x_0]$*, and they are continuous on the support with* $F(x^*) > 0$ *and* $F_e(x^*) > 0$*;*

2. H *has support* $[r_S(x^*), r_S(x_0)] = [x^* + \beta c(x^*), x_0]$ *and is continuous on the support.*

Although all agents are identical ex ante, lemma 3 shows that sellers have different production costs and buyers offer diverse prices after entering the market. This equilibrium feature stems from the assumption that sellers have private information on their production costs. We have already established that x_0 is on the support of F and F_e. If all sellers invest zero in an equilibrium, then buyers would optimally offer $p = x_0$. Anticipating this, a seller entrant could invest $c(x^*)$ without being detected and reap all the gains from the investment. So the support of F and F_e must be non-degenerate, which means that at least some sellers make positive investments.

The sellers' investment strategy has an interval support. To see this, consider any two points $x^1 < x^2$ on the support of F_e. The indifference condition and the strict convexity of $c(x)$ imply that $U(x)$ cannot be linear over the interval of $[x^1, x^2]$. By the second part of Lemma 1, some buyers should be offering prices strictly between $r_S(x^1)$ and $r_S(x^2)$. Denote one of the offered prices as $r_S(x^m)$. Buyers trade off between price and trading probability. They are willing to offer $r_S(x^m)$, which is higher than $r_S(x^1)$, only if they are compensated with a higher buying probability, which means that x^m must be on the support. Repeating this argument, F, F_e and H all have interval supports.

In equilibrium, the most efficient sellers invest $c(x^*)$ because, being the most efficient sellers, they will accept any price offered in the market and trade immediately after entry. The marginal benefit of investment thus equals 1, and they invest efficiently. In addition, there is a strictly positive measure of type x^* sellers because a buyer who offers $r_S(x^*)$ can only trade with them, though they must get a strictly positive equilibrium payoff. Moreover, there is no other mass point on the support of F and F_e because any such point would lead to a jump in the buyer's payoff in the search stage, thus contradicting the buyer's indifference condition.

On the other side of the market, buyers offer a continuum of prices with no mass point. The fact that no single price is offered by a strictly positive measure of buyers follows from the seller's indifference condition. Any such price would result in a jump in the probability of selling, which in turn leads to a kink in the function U.[14]

The preceding arguments hold even when $\beta = 0$, in which case the buyer in each pair is a monopsonist. The monopsonist uses a mixed pricing strategy because the supply function, which is endogenously determined here, is unitary elastic at any price on the support in the steady state.

[14]We base our reasoning on the assumptions about the investment cost function c, such as the continuous differentiability and the strict convexity. For a less well-behaved c, a similar argument can be used to derive the supports of the equilibrium strategies, and H may be flat in some regions or discontinuous at some points. For example, if $c'(x_0) < 0$, then $H(x_0) = Pr(p < x_0) < 1$ and $H(x_0 + \varepsilon) = 1$ for any $\varepsilon > 0$.

After establishing the supports, we can solve H and F from the indifference conditions and solve F_e from the steady state condition. The envelope condition reads,

$$U'(x) = -(1 - H(r_S(x))) + H(r_S(x))\beta U'(x).$$

The indifference condition $U'(x) = c'(x)$ solves the function H:

$$H(p) = \begin{cases} 0, \text{ if } p \in (-\infty, x^* + \beta c(x^*)), \\ \frac{1+c'(\hat{x}(p))}{1+\beta c'(\hat{x}(p))}, \text{ if } p \in [x^* + \beta c(x^*), x_0], \\ 1, \text{ if } p \in (x_0, +\infty). \end{cases} \tag{7}$$

Next, we already know from Proposition 1 that $\pi = y_0 - x_0$. Any other price on the support must yield the same expected profit. In other words,

$$(y_0 - p)F(\hat{x}(p)) + [1 - F(\hat{x}(p))]\beta(y_0 - x_0) = y_0 - x_0.$$

Then, the stationary cost distribution F can be calculated as follows:

$$F(x) = \begin{cases} 0, \text{ if } x \in (-\infty, x^*), \\ \frac{(1-\beta)(y_0-x_0)}{y_0-\beta(y_0-x_0)-x-\beta c(x)}, \text{ if } x \in [x^*, x_0], \\ 1, \text{ if } x \in (x_0, +\infty). \end{cases} \tag{8}$$

Finally, when we plug in the unique F and H into (6), the investment strategy F_e is uniquely determined. The above discussion is summarized in proposition 2. The proof is omitted in order to avoid repetition.

Proposition 2. *There is a unique steady state equilibrium with unobservable investments. In this equilibrium, sellers' reserve price strategy satisfies $r_S(x) = x + \beta c(x)$, H is given by (7), F is given by (8) and F_e is given by (6).*

4 The Effects of Unobservability

The unique equilibrium solved in the previous section shows that the seller's investment decision and the buyer's pricing strategy are different from those with observable investments, despite the fact that the payoffs are ex ante equivalent. To disentangle these results, we explore the effects of unobservability.

4.1 The Extraction-limitation Effect

Because a buyer cannot observe the paired seller's production cost, she would have to lower the price at the cost of a reduced trading probability if she wants to extract more surplus from the paired seller's investment. This trade-off limits the buyer's ability to extract surplus. We call this the ***extraction-limitation effect*** of unobservability. In equilibrium, the production cost distribution is such that ex ante identical buyers facing this trade-off are indifferent to whether they extract more or less surplus. This means that sellers (other than the least efficient ones) gain information rent when they receive high price offers, which incentivizes them to invest. Compared to the benchmark case of observable investments, these investments contribute to higher ex post surplus when trade takes place.

4.2 The Mismatch Effect

Sellers in the market have heterogeneous production costs. But because the types are acquired costly, the net gain from trading immediately, which equals $y_0 - x - \beta\pi - \beta U(x)$, must be strictly positive for any x.[15] Otherwise, a seller with a type $x < x_0$ could never trade and would not have invested $c(x) > 0$ to begin with. This property does not hold, in general, with heterogeneous agents when their types are exogenously given.

The fact that the net gain from trading immediately is always strictly positive means that if the

[15]To see this mathematically, notice that the net gain from trading immediately is $y_0 - x - \beta\pi - \beta U(x) = y_0 - \beta(y_0 - x_0) - r_S(x)$ and $r_S(x) \leq x_0$. Then the net gain from trading immediately is larger than $(1-\beta)(y_0 - x_0)$, which is strictly positive.

buyer could know the production cost of the paired seller, the two parties could always agree on a price and trade immediately. In this case, the gain from investments could be realized without any delay.

However, in the unique equilibrium with unobservable investments, both the distribution of production costs and the distribution of price offers are non-degenerate. Then, with a positive probability, a seller of a high production cost is mismatched with a buyer proposing a low price, leading to no trade, despite the fact that the net gain from trading immediately is strictly positive. To be more precise, the degree of mismatch, defined as the seller incumbents' average probability of mismatch, is measured by $\int H(r_S(x))dF(x)$. The degree of mismatch is strictly positive following Proposition 2. We call this the ***mismatch effect*** of unobservability. The mismatches cause delay in trade and erode the surplus that could be realized ex ante.

4.3 More Investments Generate No Additional Gain

The two effects of unobservability work against each other, then combine to rationalize the ex ante payoff equivalence property. Any ex post gain from investments fueled by the extraction-limitation effect is completely dissipated by the delay in trade caused by the mismatch effect.

A natural implication of this result is that alleviating the mismatch problem would restore some gains from investments. In Appendix A, we demonstrate how making the investments observable with some probability would achieve this goal and discuss related policy implications.

5 The Effects of Reducing Search Frictions

According to Proposition 1, the payoffs and the social welfare achieved in equilibrium are independent of β. That is, the level of search frictions has no impact on the welfare. This seems to contradict the robust prediction in the literature on DMBG with private information that the equilibrium outcomes become efficient as search frictions vanish. The key is that the seller entrants' type distribution is endogenous in our model, which is correlated with the trading outcomes. In this section, we examine how the trading outcomes and the entrants' type distribution change as we reduce the level of search frictions.

5.1 A higher degree of Mismatch

First, reducing the search friction results in a more aggressive price distribution. Recall that the search friction takes the form of waiting time between two meetings. As meetings become more frequent, any seller with a cost $x \in (x^*, x_0)$ has a higher chance of trading and collecting information rent per unit of time for a given price distribution, which leads to a marginal benefit of investments higher than $c'(x)$. To maintain the seller's indifference condition – that is, to discourage sellers from investing only large amounts – the probability of a seller receiving information rent per meeting should be reduced. That is, buyers must price more aggressively in the sense that a smaller proportion of them propose above any given type's reserve price, the probability of which equals $1 - H(r_S(x)) = \frac{-c'(x)(1-\beta)}{1+\beta c'(x)}$. Indeed,

$$\frac{\partial(1 - H(r_S(x)))}{\partial\beta} = \frac{(1 + c'(x))c'(x)}{(1 + \beta c'(x))^2} < 0, \text{ for any } x \in (x^*, x_0). \tag{9}$$

As the search friction vanishes, any positive probability of trade per period amounts to the marginal benefit of investment being 1. In order to keep sellers indifferent, buyers must price extremely aggressively in the sense that almost all of them must offer prices in an arbitrarily small neighborhood of the lowest reserve price $\lim_{\beta\to 1} r_S(x^*) = x^* + c(x^*)$. Indeed, $1 - H(r_S(x))$ converges to zero as β goes to 1 for any $x \in (x^*, x_0)$.

Second, the stationary cost distribution will concentrate more on high production costs. As meetings become more frequent, any buyer offering a reserve price of a given type is more likely to trade within a unit of time and would strictly prefer to offer lower prices for a given stationary cost distribution. To discourage buyers from offering only low prices, the probability of them meeting low-cost sellers should be reduced. That is, the probability $F(x)$ should be decreasing in β for any $x \in [x^*, x_0)$. An F with a larger β has first order stochastic dominance over an F with a smaller β. Indeed,

$$\frac{\partial F(x)}{\partial \beta} = \frac{(y_0 - x_0)(x - x_0 + c(x))}{[y_0 - \beta\pi - x - \beta c(x)]^2} < 0, \text{ for any } x \in [x^*, x_0). \tag{10}$$

As the time between two meetings shrinks to zero, any strictly positive probability of meeting a seller with $x < x_0$ means that a buyer can meet such a seller almost immediately after entry. For a buyer to be willing to propose x_0, $F(x)$ must converge to 0 for any $x \in [x^*, x_0)$ as β converges to 1, which can be verified by (8). That is, sellers who invested zero ex ante comprises almost all the incumbents.

The way F and H change in β implies that a smaller search friction is associated with a higher degree of mismatch.

Corollary 1. *The degree of mismatch, which is measured by $\int H(r_S(x))dF(x)$, strictly increases in β and converges to 1 as β converges to 1.*

This higher degree of mismatch explains why the equilibrium fails to converge to the first best allocation. This prediction is drastically different from what has been found in the earlier literature on DMBG. They found that trade becomes efficient when entrants' types are exogenous. That is, any trade with a positive gain takes place without any delay, which amounts to all buyers offering $r_S(x_0)$ with no mismatch in the context of our model.

5.2 Entrants Invest Efficiently

In the steady state, the cost distribution of entrants is the same as that of those who exited, which preserves the stationary cost distribution. We show that, as the search friction vanishes, the investments become efficient, i.e., F_e converges to a point mass at x^*, despite the fact that F converges to a point mass at the other end point, x_0.

In the market, seller incumbents are selected through trade as a result of the mismatch effect. The more a seller invests, the lower his reserve price is, as shown in Lemma 1, and the faster he trades and exits the market. This selection results in a less efficient stock of seller incumbents than

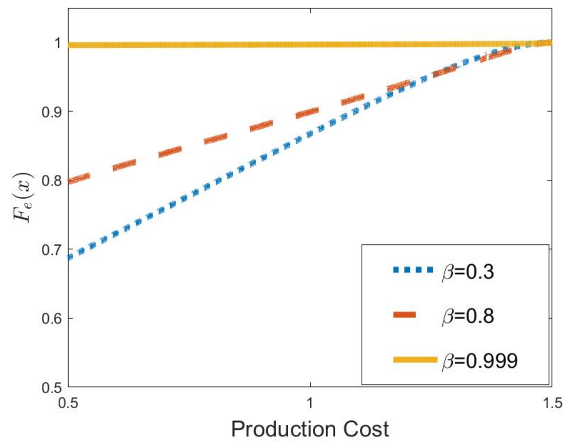

Figure 1: Seller's Investment Strategy
(*In this example:* $x_0 = 1.5$, $c(x) = \frac{1}{2}(x - x_0)^2$, $y_0 = 2.2$.)

entrants. To be more precise, F has first order stochastic dominance over F_e.[16] This selection is more prominent when buyers price more aggressively as the search friction diminishes. When the search friction vanishes, almost all underinvested incumbents (with $x > x^*$) have production costs arbitrarily close to x_0, whose probability of trade is arbitrarily close to 0, while that of the efficient sellers is always 1. This extreme selection implies that almost all exits are efficient sellers. We show that the fraction of underinvested exits converges to zero in the limit, despite the fact that the fraction of underinvested incumbents converges to 1. This is because their trading probability converges to zero. In other words, the effect of the extreme selection dominates and almost all entrants invest efficiently to replace those who have exited.

In addition, the effect of selection seems to dominate for a wide range of search frictions, as suggested by numerical examples. The vertical intercepts of the CDF curves in Figure 1 show that $F_e(x^*)$ increases in β.

[16] F has first order stochastic dominance over F_e if $F_e(x) - F(x) \geq 0$ for any $x \in [x^*, x_0]$, with strict inequality for some x. We can calculate

$$\begin{aligned} F_e(x) - F(x) &= \frac{F(x) - \int_{x^*}^{x} H(r_S(\tilde{x}))dF(\tilde{x})}{1 - \int_{x^*}^{x_0} H(r_S(\tilde{x}))dF(\tilde{x})} - F(x) \\ &= F(x)\frac{\int_{x^*}^{x_0} H(r_S(\tilde{x}))dF(\tilde{x}) - \int_{x^*}^{x} H(r_S(\tilde{x}))d\frac{F(\tilde{x})}{F(x)}}{1 - \int_{x^*}^{x_0} H(r_S(\tilde{x}))dF(\tilde{x})}. \end{aligned}$$

By the monotonicity of r_S, $F_e(x) - F(x) \geq 0$ for any $x \in [x^*, x_0]$ and the inequality is strict except for $x = x_0$.

It is worthwhile to compare the current limiting result of efficient investments with no welfare gain with the limiting case in Gul (2001) where both investments and welfare become efficient. The key difference is that Gul (2001) studies a one-to-one relationship with no repeated entry where the selection through trade is absent. Then, when investments become efficient as the time between consecutive periods shrinks, the non-investing party trades almost immediately and fully extracts the surplus. A buyer in a large market, however, meets incumbents who are much less efficient than entrants due to this selection. To extract the surplus, a buyer expects to suffer a long spell of no trade.

We summarize the comparative statics and the limiting results in this section.

Proposition 3. *As β increases, in the unique steady state equilibrium,*

1. $H(r_S(x))$ *strictly increases for any* $x \in (x^*, x_0)$*, and the distribution of price offers converges in distribution to a point mass at* $\lim_{\beta \to 1} r_S(x^*) = x^* + c(x^*)$ *as* $\beta \to 1$*;*

2. $F(x)$ *strictly decreases for any* $x \in [x^*, x_0)$*, and converges in distribution to a point mass at* x_0 *as* $\beta \to 1$.

Moreover, $F_e(x)$ *converges in distribution to a point mass at* x^* *as* $\beta \to 1$.

Proposition 3 suggests that, when estimating the efficiency of a market, both the data on the entrants' type distribution and the data on the trade efficiency are necessary. Imagine we are comparing two markets like the one studied here, with one having a larger search friction than the other. Based on the entrant's production costs alone, we may conclude that the market with a larger proportion of efficient entrants generates a higher welfare. However, this estimation neglects the interdependency of the investment strategy and trade efficiency. In fact, we have shown that there is also a higher degree of mismatch in this market and that the two markets yield the same amount of welfare.

In the next section, we extend the basic model and assume that sellers make offers with some probability in each meeting. We show that the two main conclusions of the basic model continue to hold. First, the agents' payoff as well as social welfare are independent of whether investments are observable or not. Second, the delay in trade caused by mismatch dissipates all ex post gains from any investments above the lowest investment level.

6 Random Proposers

In reality, sellers (the investing party) often also have the opportunity to make an offer. For instance, we can easily find job-wanted posts on job matching platforms initiated by freelancers that specify their desired salaries. In order to model how this affects the incentives of ex ante investments and trading outcomes, we assume that nature randomly and independently selects the seller to be the proposer with a probability of $\alpha \in (0, 1)$ and selects the buyer with the complementary probability in each pair.

6.1 Benchmark Case: Observable Investments

To figure out the effects of unobservability, let us first characterize the steady state equilibrium when investments are observable. When the seller in a pair makes an offer, he optimally proposes the buyer's reserve price $r_B = y_0 - \beta\pi$ if $r_B - x$ is larger than his discounted continuation payoff $\beta U(x)$, and proposes a higher price otherwise, which leads to no trade. When the buyer in the pair makes an offer, she optimally proposes $r_S(x) = x + \beta U(x)$ if $y_0 - x - \beta U(x)$ is larger than $\beta\pi$, and proposes a lower price otherwise, which leads to no trade. Then, trade takes place if and only if the net gain from trading immediately $y_0 - x - \beta\pi - \beta U(x)$ is weakly positive, regardless of who is proposing.

Based on this proposing strategy, the seller's search stage payoff is

$$
\begin{aligned}
U(x) &= \alpha \max\{y_0 - \beta\pi - x, \beta U(x)\} + (1-\alpha)\beta U(x) \\
&= \begin{cases} 0 & \text{for } x > y_0 - \beta\pi, \\ \frac{\alpha(y_0 - \beta\pi - x)}{1 - \beta(1-\alpha)} & \text{for } x \leq y_0 - \beta\pi. \end{cases}
\end{aligned}
\tag{11}
$$

and the seller chooses x to maximize the ex ante payoff

$$v = \max_{x \le x_0} \{U(x) - c(x)\}. \tag{12}$$

We can tell from (11) that any seller who trades must have an x lower than $y_0 - \beta\pi$, in which region $U(x) - c(x)$ is strictly concave. Then all sellers who trade must have the same production cost, denoted as $\bar{x}$, which is uniquely determined by

$$c'(\bar{x}) = \frac{-\alpha}{1 - \beta(1-\alpha)}. \tag{13}$$

Because $\alpha > 0$, $\bar{x}$ is strictly lower than x_0.

There are two possible types of equilibrium. In one type of equilibrium, which we name the **Active Equilibrium**, all seller entrants strictly prefer to invest $c(\bar{x})$ and they all trade immediately after entry. In the other type of equilibrium, which we name the **Partial-Active Equilibrium**, there are also incumbent sellers who have invested zero and never trade. Seller entrants, being indifferent between investing $c(\bar{x})$ and 0, all invest $c(\bar{x})$, which preserves the steady state distribution.

Accordingly, the buyer's payoff equals

$$\pi = \alpha\beta\pi + (1-\alpha)[F(\bar{x})(y_0 - \bar{x} - \beta U(\bar{x})) + (1 - F(\bar{x}))\beta\pi]. \tag{14}$$

Lemma 4. *With random proposers and observable investments, the steady state equilibrium exists and is unique, in which all of the seller entrants invest $c(\bar{x})$, which is uniquely defined by (13), and trade immediately after entry.*

1. *When $\alpha(y_0 - \bar{x}) \ge c(\bar{x})$, the **Active Equilibrium** holds, in which $F(\bar{x}) = 1$ and all agents trade immediately after entry. In equilibrium, $v = \alpha(y_0 - \bar{x}) - c(\bar{x})$, $\pi = (1-\alpha)(y_0 - \bar{x})$ and $s = y_0 - \bar{x} - c(\bar{x})$.*

2. *When $\alpha(y_0 - \bar{x}) < c(\bar{x})$, the **Partial-Active Equilibrium** holds, in which $F(x) = \frac{\alpha(y_0-\bar{x})}{\beta(1-\alpha)c(\bar{x})} - \frac{1-\beta(1-\alpha)}{\beta(1-\alpha)} < 1$ for any $x \in [\bar{x}, x_0)$, $F(x_0) = 1$ and type x_0 seller incumbents never trade. In*

equilibrium, $v = 0$ *and* $\pi = s = \frac{\alpha(y_0-\bar{x})-(1+\alpha\beta-\beta)c(\bar{x})}{\alpha\beta}$.

In order to see that the two types of equilibrium indeed exist with the right parameters, consider the following example, where $c(x) = x^2 - 2x + 1$, $x_0 = 1$, $\alpha = 0.1$ and $\beta = 0.9$.[17] The production cost $\bar{x}$ equals $1 - \frac{1}{3.8}$ according to (13). Then the condition $\alpha(y_0 - \bar{x}) \geq c(\bar{x})$ is equivalent to $y_0 \geq 10 \times (1 - \frac{1}{3.8})^2 - 19 \times (1 - \frac{1}{3.8}) + 10 \approx 1.429$. Therefore, the Active Equilibrium holds when $y_0 > 1.43$ and the Partial-Active Equilibrium holds when $y_0 \in (1, 1.42)$.

The dynamics of the Partial-Active Equilibrium need further clarification. When $\alpha(y_0 - \bar{x}) < c(\bar{x})$ holds, a type $\bar{x}$ seller would earn a negative ex ante payoff if buyers were to trade with no delay and consequently have a low reserve price r_B. Then, in order to incentivize investment, there must be some type x_0 incumbents who entered in the past and could never trade, causing a delay for buyers.[18] The measure of type x_0 incumbents is such that it causes just enough delay and lowers π by just the right amount to make seller entrants indifferent between investing $c(\bar{x})$ and 0.

6.2 The Steady State Equilibrium

Let us now turn to the steady state equilibrium with unobservable investments. The seller's search stage payoff is[19]

$$U(x) = \begin{cases} 0 & \text{for } x > y_0 - \beta\pi, \\ \frac{\alpha(y_0-\beta\pi-x)+(1-\alpha)\int_{r_S(x)}(p-x)dH(p)}{1-\beta(1-\alpha)H(r_S(x))} & \text{for } x \leq y_0 - \beta\pi. \end{cases}$$

and the ex ante payoff equals

$$v = \max_{x \leq x_0} \{U(x) - c(x)\}.$$

[17] We can verify that function c satisfies all the regularity conditions assumed.

[18] Before reaching the steady state, seller entrants randomize between investing $c(\bar{x})$ and zero, and the non-investing sellers never trade and stay in the market. When the market accumulates enough type x_0 sellers, the economy reaches the steady state and all of the entrants invest $c(\bar{x})$.

[19] We can show in a similar way that a seller never trades if the net gain from trading immediately is negative.

Then following the same logic as in the case of observable investments, with any x on the support of F_e, the net gain from trading immediately is positive.

The highest production cost on the support of F_e, although unobservable, is known in the equilibrium. Here, this seller with the highest production cost is either fully extracted or unable to trade when the buyer in the pair proposes. Hence his search stage payoff will be the same as in (11) when investments are observable. The payoff of a buyer who offers this seller's reservation price will be the same as in (14) when investments are observable. Therefore, the upper bound of the support of F_e is still $\bar{x}$, and $F(\bar{x})$ coincides with the observable benchmark. The ex ante payoff equivalence property in the basic model continues to hold regardless of the investors' bargaining power.

Proposition 4. ***(Ex Ante Payoff Equivalence)*** *In the steady state equilibrium with random proposers and unobservable investments, the buyers' and sellers' ex ante payoffs coincide with the payoffs in the observable benchmark.*

Moreover, following the same logic as in the basic model, both the sellers and the buyers are using mixed strategies. The CDF F_e and H are continuous and have support $[x^*, \bar{x}]$ and $[r_S(x^*), r_S(\bar{x})]$, respectively, with $F_e(x^*) > 0$. Combined with the payoff equivalence result, this means that, although all entrants invest more than $c(\bar{x})$, any extra investment beyond $c(\bar{x})$ does not contribute to anyone's ex ante payoff. Like in the basic model, this zero-gain result stems from the extraction-limitation effect and the mismatch effect working against each other. We elaborate below.

A seller's investment can be decomposed into two parts, $c(\bar{x})$ and beyond, according to the source of incentives. He is willing to invest $c(\bar{x})$ because he proposes in each meeting with probability α. In the basic model, this part of the investment is zero because $\alpha = 0$. His investment beyond $c(\bar{x})$ is induced by the extraction-limitation effect. On the other hand, the double-mixing strategy means that the mismatch effect continues to play a role as long as $\alpha < 1$ and $\beta < 1$: when the buyer in a pair proposes with probability $1 - \alpha$, there is a strictly positive probability that the proposed price will be lower than (i.e., mismatches with) the reserve price of the seller, although the net gain from trading immediately will be strictly positive. Then the no-gain property in the basic model is extended: any investment fueled by the extraction-limitation effect (those beyond $c(\bar{x})$) adds no ex ante gain due to the mismatch effect.

To summarize, the payoff equivalence and the no-gain property are not knife-edge results which only hold when sellers have no bargaining power. Instead, the two properties hold independent of the bargaining power allocation.

6.3 The Effects of Reducing Search Frictions

In this section, we investigate the effects of reducing search frictions. To highlight the fact that $\bar{x}$ and s depend on α and β, we use $\bar{x}(\alpha, \beta)$ and $s(\alpha, \beta)$ to denote the two.

Corollary 2. *In the steady state equilibrium with random proposers and unobservable investments, when $\alpha > 0$, as β increases, $\bar{x}(\alpha, \beta)$ decreases and converges pointwise (but not uniformly) to x^*, $s(\alpha, \beta)$ increases and converges pointwise (but not uniformly[20]) to s^*.*

We know that the lowest investment amount $c(\bar{x})$ is motivated by the opportunity to propose. A seller proposes in each meeting with a positive probability. As meetings arise more frequently, sellers are more likely to propose within a unit of time, which motivates them to invest more. Moreover, trade takes place without delay when a seller proposes. Therefore, the mismatch problem is mitigated as the search friction diminishes, which leads to a higher welfare. As we can see from this argument, unobservability of the investment plays no role in improving social welfare.

The fact that $s(\alpha, \beta) > s_0$ when $\alpha > 0$ demonstrates, again, the importance of using the data on both the trade efficiency and the entrants' type distribution when evaluating the market efficiency. Consider two markets: one where β is close to 1 and $\alpha = 0$, and the other where β is small and α is small but positive, such that sellers entering the market are more efficient, on average, in the

[20]It is easy to verify that $\bar{x}(\alpha, \beta)$ and $s(\alpha, \beta)$ are continuous in α. The convergence of $\bar{x}$ and s as β increases to 1 is not uniform, because there does not exist a single $\bar{\beta}(\varepsilon)$, such that $\bar{x}(\alpha, \beta) - x^* < \varepsilon$ or $s^* - s(\alpha, \beta) < \varepsilon$ for any $\alpha > 0$ and any $\beta > \bar{\beta}(\varepsilon)$. In particular, when α is arbitrarily close to 0, the meetings need to arrive arbitrarily frequently (the speed of which depends on α) so that a seller has the chance to make an offer almost immediately after entry.

first market. The data on the entrants' production costs alone suggests that the first market would yield a higher social welfare, but that the second market is actually more efficient.

6.4 Random Proposers and Costly Entry

In reality, entering a market could be costly. In the online appendix, we consider a scenario where a seller decides, first, whether to pay a fixed cost to enter and, if he does, how much he will invest in order to lower his production costs. We incorporate this endogenous entry decision into the setting with random proposers, and then explore how the seller's extensive margin of that entrance decision interacts with the investment incentives and trade efficiency.[21] We show that, with positive search frictions, there is either redundant or insufficient entry unless the bargaining power takes on a particular value that depends on the investment cost function.[22] When the search friction vanishes, however, both the investment strategy and the entry decision become efficient as long as the seller's probability of proposing is strictly positive. This means that the convergence results in the setting with random proposers and exogenous entry are robust for this alternative assumption of costly entry.

In Appendix A, we explore two extensions of the basic model and discuss their policy implications. First, when buyers can also invest before entering the market, unobservability results in buyers using a mixed investment strategy and thus underinvesting even when they have full bargaining power. We also show that when buyers invest, it alleviates the mismatch problem. In the second extension, we demonstrate that, if the investment is randomly observable, the mismatch problem

[21] When $\alpha = 0$, any positive entry costs would lead to an empty market in the steady state because sellers are fully extracted. The entry decision is non-trivial in the setting with random proposers. On the other hand, the basic model can be seen as a market with zero entry cost in which there is a continuum of equilibria with different measures of seller entrants. We focus on the equilibrium where the measure of buyer- and seller-entrants is equal, which maximizes the social welfare when the search friction vanishes, as we show in the online appendix.

[22] This discussion on efficient bargaining power is related to the literature on optimal trading protocol with search friction, such as Hosios (1990) and Delacroix and Shi (2018). The common emphasis is on how trading protocols influence the efficiency through the agents' participation decisions. The difference between Hosios (1990) and our paper is that the optimal bargaining power in our model depends not only on the matching function, but also on the investment cost function. Delacroix and Shi (2018) study how entry costs and the costs of organizing trade determine which side should organize the trade to maximize the surplus in a directed search model. Similarly, we highlight how costs of entry and the cost of investment determine the optimal bargaining protocol.

is alleviated and buyers gain a higher payoff. This implies that social welfare is non-monotonic in observability and that making investments randomly observable improves social welfare. In the online appendix, we study two more extensions. First, we show that, if buyers can costly verify the paired seller's production cost, they will either randomize between verifying and not verifying, or will not verify. Therefore, the agents' payoffs are the same as in the basic model. Second, we discuss how the ex ante payoff equivalence property and the mixed strategy equilibrium might survive when sellers are ex ante heterogeneous.

7 Conclusion

This paper examines the effects of private information when market participants can invest in order to change their characteristics. In this case, the entrants' type distribution is correlated with the trading outcomes rather than exogenously given. We show that, although agents invest a positive amount, the equilibrium payoffs will be the same as if investments were observable and agents invested nothing. Unobservability has two effects: the extraction-limitation effect, which motivates investments, and the mismatch effect, which causes trade inefficiency. The two opposing effects exactly cancel each other out given any search friction. Even when the search friction vanishes and sellers invest almost efficiently, social welfare does not converge to the first best level in the basic model due to the persistent mismatch effect.

As for future research, one possible direction to investigate would be the optimal bargaining power allocation when agents on both sides make investments that are unobservable. This problem is non-trivial because the allocation needs to strike a balance between minimizing mismatches and maximizing both sides' investment incentives.[23] We could also extend the analysis to a scenario in which a seller's investments not only benefit himself but also raise the valuation of the paired buyer. This additional feature means that, when making the price offers, a buyer cares not only about the probability of trade, as in the current model, but also her seller's type.

[23] More detailed discussion could be found in Appendix A.A.

Appendices

A Other Extensions and Policy Implications

In this section, we discuss two other extensions and then move on to policy implications for the basic model and the extensions.

Appendix A.A Other Extensions

Two-sided Investments In the first extension, we assume that both sides of the market can invest before entry. We demonstrate that buyers will underinvest even if they have full bargaining power as a consequence of unobservability and that the mismatch problem is mitigated when buyers can also invest.

In particular, assume that a buyer can raise her valuation to y with investments $e(y)$, where $e(y_0) = e'(y_0) = 0$, and $e(y)$ is strictly increasing, strictly convex and continuously differentiable for any $y > y_0$.[24] The observability of the buyer's investment is irrelevant, because we retain the assumption that buyers have full bargaining power, as in the basic model.

Firstly, we can see that the ex ante payoff equivalence property remains true in this extension, implying that this property is not restricted to one-sided investments. When investments are observable, sellers are fully extracted at the search stage and hence invest zero ex ante. Buyers offer x_0 and trade immediately after entry, meaning that their marginal benefit of investment is 1, and they will therefore invest $e(y^*)$, which satisfies $e'(y^*) = 1$. When investments are unobservable, the sellers who have invested the lowest amount are either fully extracted or find the price too low to accept, which makes their search stage payoff 0 and the ex ante investments 0. Buyers who propose the corresponding reserve price x_0 are offering the highest price and hence trade immediately after entry, inducing them to invest $e(y^*)$. Then sellers receive zero payoff ex ante and buyers

[24] This assumption excludes any complementarity between investments, which greatly simplifies the analysis but precludes some interesting and relevant scenarios. For instance, the surplus from trade could be supermodular in x and y, in which case the marginal benefit of the buyer's investment increases in the seller's investment. We will leave the analysis of more general cases for future work.

receive payoff $y^* - x_0 - e(y^*)$ ex ante with either observable or unobservable investments.

In the analysis that follows, we focus on the buyer's investment strategy and the steady state valuation distribution in order to highlight the new results. Use $G_e(y)$ to denote the probability that a buyer entrant invests weakly less than $e(y)$ and use $G(y)$ for the probability that a buyer incumbent has a valuation weakly lower than y.

Proposition 5. *In the unique steady state equilibrium with two-sided investments,*

1. *G and G_e are continuous and have support $[\underline{y}, y^*]$, where $G(\underline{y}) = G_e(\underline{y}) = 0$, $\underline{y} < y^*$ and it is uniquely determined by*

$$y^* - x_0 - e(y^*) = [\underline{y} - x^* - \beta c(x^*)]e'(\underline{y}) - e(\underline{y}).$$

   The social welfare and the buyer's ex ante payoff equals $y^* - x_0 - e(y^*)$, and sellers get an ex ante payoff of 0.

2. *As $\beta \to 1$, $\lim_{\beta\to 1} \underline{y} < y^*$ and $\lim_{\beta\to 1} G(\underline{y}) = \lim_{\beta\to 1} G_e(\underline{y}) = 1$.*

In the unique steady state, the support of G_e is non-degenerate, meaning that buyers underinvest even when they have full bargaining power. This holds even when the search friction vanishes, as stated in the second part of Proposition 5. The reason for underinvesting stems from the presence of diverse price offers, which, as we have shown, must be true when investments are unobservable. Buyers who offer a lower price tend to trade more slowly compared to those who offer a higher price. They have a smaller marginal benefit of investment and therefore invest less. When the search friction vanishes, almost all buyers offer the lowest reserve price and experience delay in trade. Consequently, almost all buyers invest $e(\underline{y}) < e(y^*)$.

Another interesting feature here is that the mismatch problem is mitigated when buyers also invest. The welfare is $y^* - x_0 - e(y^*)$ while almost all buyers invest less than $e(y^*)$. It must be the case that some of the gain from the sellers' investments is realized ex ante because of a lower degree of mismatch. To see why, notice that buyers have higher ex post gains from trade when they also invest and have a stronger incentive to speed up trade. To keep buyers indifferent, the cost distribution of incumbent sellers must be more efficient so that buyers are more likely to trade in a meeting with a given price. This leads to a lower degree of mismatch.

A natural question to ask is, what would happen if both sides could propose and make investments that are unobservable to their partners? Agents would continue to play mixed investment strategies due to unobservability. Otherwise, suppose all sellers were to invest the same amount, which is less than $c(x^*)$, and all buyers were to invest the same amount as well. Then all buyers would offer the same price and have the same reserve price. A seller would optimally deviate to invest $c(x^*)$ and obtain all the gains. Anticipating this, buyers would offer the price $r_S(x^*)$ instead and extract all the surplus when they make their offers. This would undermine the sellers' incentive to invest, so they would invest less than $c(x^*)$, which brings us back to the beginning of this argument. However, a complete characterization of the equilibrium requires considerable extra effort and is beyond the scope of this paper. To see where the difficulties lie, notice that, in the extension of two-sided investments, the ex ante payoffs can be obtained independent of the stationary distributions. Then we can derive the stationary distributions based on the ex ante payoffs. In the scenario considered here, however, the ex ante payoffs and the stationary distributions in equilibrium are linked and cannot be determined separately. Consider the least efficient seller, whose payoff was used to calculate all sellers' ex ante payoff in the previous extension. The highest reserve price that he proposes will only be accepted by the buyers with the highest valuation. His payoff will depend on the percentage of buyers who are investing the highest amount in the market as well as these buyers' reserve price, which are equilibrium objects. We leave more detailed analysis to future work.

When Investments are Observable with a Positive Probability In the second extension, we show that the buyers' payoff and social welfare will be higher when investments are randomly observable. Combined with the payoff equivalence property of the basic model, this means that social welfare is not monotonic in observability.

The discussion in the basic model suggests that if the mismatch problem can be alleviated, then the investments should be able to improve social welfare. To see this, assume that the investments are observable with probability $q \in (0, 1)$ in each meeting independently. Following the logic in the basic model, sellers get an ex ante payoff of zero and the production costs in the market range from $\underline{x} < x_0$ to x_0. A type $\underline{x}$ seller receives information rent when investments are unobservable, and is fully extracted otherwise. That is, $U(\underline{x}) = q\beta U(\underline{x}) + (1-q) \int (p-\underline{x}) dH(p)$. The condition $c'(\underline{x}) = \frac{-(1-q)}{1-q\beta}$ then uniquely determines $\underline{x}$, and $\underline{x} < x_0$. Buyers offer a range of prices when investments are unobservable. Among them, some offer $r_S(x_0) = x_0$ when investments are unobservable and $r_S(x) = x + \beta c(x)$ if the production cost is revealed to be x. Accordingly, the buyer's payoff can be calculated as

$$\pi = (y_0 - x_0)(1 - q) + q \int_{\underline{x}}^{x_0} [y_0 - x - \beta c(x)] dF(x).$$

We can verify that the buyer's payoff, which equals the social welfare, is strictly larger than s_0. This is because, when investments are observable, there is no mismatch. Then, the gain from investments, which is strictly positive, is realized without any delay, which is reflected in the second term of π.

Appendix A.B Policy Implications

It might seem intuitive that reducing the search friction would ease trade and thus raise social welfare. Our basic model, however, demonstrates that, when sellers can make ex ante investments but have no bargaining power, reducing the search friction actually leads to a higher degree of mismatch and thus does not improve social welfare.

We know, however, that seller entrants indeed invest positive amounts and generate ex post gains. The key is to mitigate the mismatch problem so that these ex post gains can be realized as early as possible. Our analysis suggests three measures.

First, making investments randomly observable can improve trade efficiency. For example, an authority could verify the production cost of a seller and provide this information to a buyer with some probability. In the online appendix, we show in the extension about costly verification that allowing buyers to verify at a cost does not change social welfare, because buyers must be indifferent between verifying and not verifying in equilibrium. So, indeed, some benevolent third party is needed to collect the information in order to improve social welfare. Second, encouraging buyers to invest may also alleviate the mismatch problem. On the other hand, with these two measures, sellers still get an ex ante payoff of zero because they have no bargaining power. Therefore, if the policy maker aims to improve social welfare and raise the sellers' payoff at the same time, he or she could consider increasing the sellers' bargaining power. If the policy maker is able to reduce the search friction simultaneously, then social welfare would further increase and converge to the first best as the search friction vanishes.

B Proofs

Proof of Lemma 1

1. A type x seller can always adopt the reserve price of a type $x+\epsilon$ seller, for some $\epsilon > 0$. Function U is therefore strictly decreasing in x.

We know

$$U(x) = (E(p \mid p \geq r_S(x)) - x)(1 - H(r_S(x))) + H(r_S(x))\beta U(x).$$

After multiplying both sides by β and adding x, the left-hand side becomes $r_S(x)$. We re-arrange the equation to have:

$$(1-\beta)r_S(x) + \beta \left[r_S(x) - E(p \mid p \geq r_S(x))\right](1 - H(r_S(x))) = (1-\beta)x.$$

The left-hand side strictly increases in $r_S(x)$ and the right-hand side strictly increases in x, which implies that $r_S(x)$ strictly increases in x.

Denote the highest price that a buyer offers as $r_S(\tilde{x})$. Plugging in any $x \geq \tilde{x}$, equation (1) becomes $U(x) = \beta U(x)$, which is equivalent to $U(x) = 0$, implying that U is continuous at $x \geq \tilde{x}$. For $x < \tilde{x}$, U can only have downward jumps because it decreases in x. A downward jump of U is equivalent to a downward jump of r_S at the corresponding point. This contradicts the earlier conclusion that r_S strictly increases in x. Then both U and r_S are continuous.

2. Consider any $x_l, x_h \in (x^1, x^2)$. For any $\lambda \in (0, 1)$, denote $\lambda x_l + (1-\lambda)x_h$ as x_λ. When $H(r_S(x^2)) - H(r_S(x^1)) - Pr(p = r_S(x^1)) = 0$, $H(r_S(x_l)) = H(r_S(x_h)) = H(r_S(x_\lambda))$ and $E(p \mid p \geq r_S(x_l)) = E(p \mid p \geq r_S(x_h)) = E(p \mid p \geq r_S(x_\lambda))$. Then

$$\begin{aligned}\lambda U(x_l) + (1-\lambda)U(x_h) =& (E(p \mid p \geq r_S(x_\lambda)) - x_\lambda)(1 - H(r_S(x_\lambda))) \\ &+ H(r_S(x_\lambda))\beta\left[\lambda U(x_l) + (1-\lambda)U(x_h)\right].\end{aligned}$$

That is, $U(x_\lambda) = \lambda U(x_l) + (1-\lambda)U(x_h)$. The function U is linear over the interval $[x^1, x^2]$.

Proof of Lemma 2 and Proposition 1

According to (4), the buyer's payoff decreases in p for any $p > r_S(\bar{x})$. Therefore, buyers never offer a price above $r_S(\bar{x})$. Then $U(\bar{x}) = 0$ and $U(\bar{x}) - c(\bar{x}) = -c(\bar{x})$, which implies that $\bar{x} = x_0$ and $r_S(\bar{x}) = \bar{x} + \beta U(\bar{x}) = x_0$. Then $v = U(\bar{x}) - c(\bar{x}) = 0$.

The price $r_S(x_0) = x_0$ must be on the support of H. Otherwise, suppose the highest price on the support is $r_S(\tilde{x})$, where $\tilde{x} < x_0$ because no buyer would offer a price higher than x_0. Then a type $\tilde{x}$ seller would earn zero payoff on the search stage based on (1) and a negative ex ante payoff of $-c(\tilde{x})$, which implies that $\tilde{x}$ is not on the support of F. In this case, buyers should not offer $r_S(\tilde{x})$. This leads to a contradiction. Then $\pi = y_0 - x_0 = s_0$ and $s = v + \pi = s_0$.

Proof of Lemma 3

We first note that if a price offer $p > r_S(\underline{x})$ is on the support of H, then $\hat{x}(p)$ must be on the support of F and F_e. Otherwise, denote the highest production cost on the support that is lower than $\hat{x}(p)$ as x'. A buyer who offers price p can lower the price to $r_S(x')$ without affecting the probability of trade. Also, if $p = r_S(\underline{x})$ is on the support of H, then F and F_e must have a point mass at $\underline{x}$. It is because a buyer who offers $r_S(\underline{x})$, which is only accepted by type $\underline{x}$ sellers, would get a payoff of zero if $F(\underline{x}) = 0$ and obtain a higher profit by deviating to the price of x_0. Now suppose that there are p_1, p_2 on the support of H, such that any $p \in (p_1, p_2)$ is not offered. Since p_1 and p_2 are on the support, $x_i \equiv \hat{x}(p_i)$, $i = 1, 2$ must be on the support of F and F_e. By the second part of Lemma 1, the function U is linear over the interval $[x_1, x_2]$. On the other hand, $c(x)$ is strictly convex. This means that $U(x_1) - c(x_1) = U(x_2) - c(x_2) < U(x) - c(x)$ for any $x \in (x_1, x_2)$, which contradicts the indifference condition. Therefore, H has an interval support. By the continuity of r_S, F and F_e also have an interval support.

The lower bound of the price offers never goes below the reserve price of the most efficient seller, i.e., $H(r_S(\underline{x})) = 0$, because, otherwise, a buyer offering the lowest price would never trade, preferring to offer x_0 and earn $y_0 - x_0 > 0$. Plugging $H(r_S(\underline{x})) = 0$ into the envelope condition $U'(x) = -(1 - H(r_S(x))) + H(r_S(x))\beta U'(x)$ at $x = \underline{x}$, it becomes $U'(\underline{x}) = -1$. Then the indifference condition would imply $U'(\underline{x}) = c'(\underline{x}) = -1$, meaning that $\underline{x} = x^*$.

Hence, the support of F and F_e is $[x^*, x_0]$ and the support of H is $[r_S(x^*), r_S(x_0)]$.

Next, we show that H has no atom. Function U is differentiable at any x on the support, because 1) F strictly increases over the support, 2) $U(x) - c(x) = 0$ and 3) $c(x)$ is differentiable. This implies that function r_S and $\hat{x}$ are differentiable at any x and p on the support. Hence, we can solve $H(p)$ from the equilibrium condition $U'(x) = c'(x)$ as follows:

$$H(r_S(x)) = \frac{1 + c'(x)}{1 + \beta c'(x)} \Rightarrow H(p) = \frac{1 + c'(\hat{x}(p))}{1 + \beta c'(\hat{x}(p))}.$$

It is straightforward to verify that H has no atom.

Finally, we show that F and F_e have no point mass at any $x > x^*$. Suppose $x \in (x^*, x_0]$ is a mass point. Then there exists an $\varepsilon > 0$, such that buyers do not offer any price $p \in (r_S(x - \varepsilon), r_S(x))$, because, by raising the price slightly to $r_S(x)$, buyers enjoy a discontinuous upward jump of the trading probability. This contradicts the earlier conclusion that H has an interval support.

Proof of Corollary 1

Take any $\beta^1, \beta^2 \in (0,1)$ with $\beta^1 < \beta^2$. Denote the corresponding price strategy and the stationary cost distribution as $H^i(r_S^i(x))$ and $F^i(x)$ for $i = 1, 2$. We can calculate

$$\int H^2(r_S^2(x))dF^2(x) - \int H^1(r_S^1(x))dF^1(x)$$
$$= \left(\int H^2(r_S^2(x))dF^2(x) - \int H^1(r_S^1(x))dF^2(x)\right) + \left(\int H^1(r_S^1(x))dF^2(x) - \int H^1(r_S^1(x))dF^1(x)\right)$$

The first parenthesis is strictly positive following (9). The second parenthesis is also strictly positive because of (10) and the fact that $H(r_S(x))$ strictly increases in x. This proves that the degree of mismatch strictly increases in β.

That $\int H(r_S(x))dF(x)$ converges to 1 when β converges to 1 follows from the fact that $H(r_S(x_0)) = 1$ and that $F(x)$ converges in distribution to a point mass at x_0.

Proof of Proposition 3

The first two parts of the proposition have been proven in the main text. We only need to show that $F_e(x^*)$ approaches 1 when β converges to 1. By (6),

$$F_e(x^*) = [1 + \frac{1}{F(x^*)}\int_{x^*}^{x_0}(1 - H(r_S(x)))f(x)dx]^{-1} \equiv [1 + A]^{-1}. \tag{15}$$

For any $\epsilon_1, \epsilon_2 > 0$ that satisfy $\epsilon_1 + \epsilon_2 < x_0 - x^*$, by the Median Value Theorem, there exist $x_1 \in (x^*, x^* + \epsilon_1)$, $x_2 \in (x^* + \epsilon_1, x_0 - \epsilon_2)$, and $x_3 \in (x_0 - \epsilon_2, x_0)$, such that

$$\begin{aligned} A =& \frac{1}{F(x^*)}\left[\int_{x^*}^{x^*+\epsilon_1}(1 - H(r_S(x)))f(x)dx + \int_{x^*+\epsilon_1}^{x_0-\epsilon_2}(1 - H(r_S(x)))f(x)dx + \int_{x_0-\epsilon_2}^{x_0}(1 - H(r_S(x)))f(x)dx\right] \\ =& [1 - H(r_S(x_1))]\frac{F(x^* + \epsilon_1) - F(x^*)}{F(x^*)} + [1 - H(r_S(x_2))]\frac{F(x_0 - \epsilon_2) - F(x^* + \epsilon_1)}{F(x^*)} \\ &+ [1 - H(r_S(x_3))]\frac{1 - F(x_0 - \epsilon_2)}{F(x^*)}. \end{aligned} \tag{16}$$

First, fix any ϵ_1 and ϵ_2, and consider the second term of (16). The term $F(x_0 - \epsilon_2) - F(x^* + \epsilon_1)$ can be expressed as

$$\frac{(y_0 - x_0)(-(x^* + \epsilon_1) - \beta c(x^* + \epsilon_1) + (x_0 - \epsilon_2) + \beta c(x_0 - \epsilon_2))}{(y_0 - x_0 + \frac{\epsilon_2 - \beta c(x_0 - \epsilon_2)}{1-\beta})(y_0 - \beta(y_0 - x_0) - (x^* + \epsilon_1) - \beta c(x^* + \epsilon_1))}.$$

While all other terms are bounded for any $\beta \in [0, 1)$, given any ξ, the term $\frac{\epsilon_2 - \beta c(x_0 - \epsilon_2)}{1-\beta} > 1/\xi$ when $\beta > \frac{1-\xi\epsilon_2}{1-\xi c(x_0-\epsilon_2)} \in (0, 1)$. This implies that $1 - F(x_0 - \epsilon_2) > F(x_0 - \epsilon_2) - F(x^* + \epsilon_1)$ when β is close enough to 1. At the same time,

$$[1 - H(r_S(x_2))] - [1 - H(r_S(x_3))] = (1 - \beta)\left(\frac{-c'(x_2)}{1 + \beta c'(x_2)} - \frac{-c'(x_3)}{1 + \beta c'(x_3)}\right).$$

The terms inside the second parentheses of the right-hand side are bounded for any given ϵ_1, ϵ_2 and β. Therefore, $[1 - H(r_S(x_2))] - [1 - H(r_S(x_3))]$ is close to 0 when β is close to 1. Combined, $[1 - H(r_S(x_2))][F(x_0 - \epsilon_2) - F(x^* + \epsilon_1)] < [1 - H(r_S(x_3))][1 - F(x_0 - \epsilon_2)]$ when β is close enough to 1, which is equivalent to the second term being smaller than the third term of (16).

Second, the third term of (16) is smaller than

$$[1 - H(r_S(x_3))]\frac{1 - F(x^*)}{F(x^*)} = \frac{-c'(x_3)(x_0 - x^* - \beta c(x^*))}{(y_0 - x_0)(1 + \beta c'(x_3))}.$$

When ϵ_2 is arbitrarily small, x_3 is arbitrarily close to x_0, which implies that the third term is arbitrarily close to 0 independent of β.

Next, by the continuity of F, when ϵ_1 is arbitrarily small, $\frac{F(x^*+\epsilon_1)}{F(x^*)}$ is arbitrarily close to 1. Because $1 - H(r_S(x_1))$ is bounded, the first term is arbitrarily close to 0 independent of β when ϵ_1 is arbitrarily close to 0.

Finally, based on the conclusion in the first step that the second term is smaller than the third term for any given ϵ_1 and ϵ_2 when β is large enough, we can conclude that the second term is arbitrarily close to 0 when β is large enough.

Then, according to (15), $F_e(x^*)$ converges to 1 when β converges to 1.

Proof of Lemma 4

What remains to be shown are the payoffs and the stationary cost distribution in the two types of equilibrium, as well as the condition of primitives under which each type of equilibrium holds.

Active Equilibrium. Plug $F(\bar{x}) = 1$ into (14), and the ex ante payoffs can be solved from (11) to (14) as $v = \alpha(y_0 - \bar{x}) - c(\bar{x})$ and $\pi = (1-\alpha)(y_0 - \bar{x})$, in which $\bar{x}$ is uniquely determined by (13). The welfare $s = v + \pi = y_0 - \bar{x} - c(\bar{x})$. For this equilibrium to arise, sellers must have an incentive to invest, i.e., $\alpha(y_0 - \bar{x}) \geq c(\bar{x})$ should hold.

Partial-Active Equilibrium. In the Partial-Active Equilibrium, seller's ex ante payoff $v = 0$ because they are indifferent between investing $c(\bar{x})$ and 0. At the same time, $v = \frac{\alpha}{1+\alpha\beta-\beta}(y_0 - \bar{x} - \beta\pi) - c(\bar{x})$ given π. Then π can be solved from the condition $v = 0$ as

$$\pi = \frac{\alpha(y_0 - \bar{x}) - (1 + \alpha\beta - \beta)c(\bar{x})}{\alpha\beta}.$$

The welfare $s = v + \pi = \pi$. We also know from the buyer's value function that

$$\begin{aligned}\pi =& \alpha\beta\pi + (1-\alpha)[F(\bar{x})(y_0 - \bar{x} - \beta U(\bar{x})) + (1 - F(\bar{x}))\beta\pi] \\ =& \alpha\beta\pi + (1-\alpha)[F(\bar{x})(y_0 - \bar{x} - \beta c(\bar{x})) + (1 - F(\bar{x}))\beta\pi].\end{aligned}$$

Equating the two π's, $F(\bar{x})$ can be solved as:

$$F(\bar{x}) = \frac{\alpha(y_0 - \bar{x})}{\beta(1-\alpha)c(\bar{x})} - \frac{1 - \beta(1-\alpha)}{\beta(1-\alpha)}.$$

Because all entrants invest $c(\bar{x})$, $F(x) = F(\bar{x})$ for any $x \in [\bar{x}, x_0)$. For this to be an equilibrium, $F(\bar{x})$ should be larger than 0 and smaller than 1. The former is equivalent to $\frac{\alpha}{1-\beta(1-\alpha)} > \frac{c(\bar{x})}{y_0 - \bar{x}}$, which always hold because (13) implies $\frac{\alpha}{1-\beta(1-\alpha)} > \frac{c(\bar{x})}{x_0 - \bar{x}} > \frac{c(\bar{x})}{y_0 - \bar{x}}$. The latter requires $\alpha(y_0 - \bar{x}) < c(\bar{x})$.

In sum, the above discussion shows that the Active Equilibrium emerges and is unique when $\alpha(y_0 - \bar{x}) \geq c(\bar{x})$. Otherwise, the Partial-Active Equilibrium emerges and is unique.

Proof of Proposition 4

Because $\bar{x}$ is on the support of F_e, agents' payoffs can be calculated based on the payoff of a seller who invests $c(\bar{x})$ and that of a buyer who offers $r_S(\bar{x})$. Their payoffs are independent of the distributions H and F and are the same as those in the benchmark case with observable investments.

Proof of Corollary 2

According to (13), for any given $\alpha > 0$, $\bar{x}(\alpha, \beta)$ strictly decreases in β because $\frac{-\alpha}{1-\beta(1-\alpha)}$ strictly decreases in β and the function c is strictly convex. In addition, $\frac{-\alpha}{1-\beta(1-\alpha)}$ converges to -1 as β converges to 1. Therefore, $\bar{x}(\alpha, \beta)$ converges pointwise to x^*. We can also calculate that $1 - c'(\overline{x}) < \varepsilon$ is equivalent to $\beta > \frac{1}{1-\alpha} - \frac{\alpha}{(1-\alpha)(1-\varepsilon)}$. There does not exist a $\bar{\beta} < 1$, such that $\bar{x}(\alpha, \beta) - x^*$ is arbitrarily small whenever $\beta \in (\bar{\beta}, 1)$ for any $\alpha > 0$. Therefore, $\bar{x}(\alpha, \beta)$ does not uniformly converge to x^*.

In the Active Equilibrium,

$$s(\alpha, \beta) = y_0 - \bar{x}(\alpha, \beta) - c(\bar{x}(\alpha, \beta)),$$

which is decreasing in $\bar{x}(\alpha, \beta)$. Therefore, for any given $\alpha > 0$, $s(\alpha, \beta)$ strictly increases in β and converges pointwise to s^*. In the Partial-Active Equilibrium,

$$s(\alpha, \beta) = \frac{1}{\beta}[y_0 - \bar{x}(\alpha, \beta) - \frac{1 + \alpha\beta - \beta}{\alpha} c(\bar{x}(\alpha, \beta))].$$

For any given $\alpha > 0$, $s(\alpha, \beta)$ strictly increases in β because (taking into account that $\bar{x}$ decreases in β)

$$\frac{\partial s(\alpha, \beta)}{\partial \beta} \propto c(\bar{x}) - \alpha(y_0 - \bar{x}) > 0.$$

As β converges to 1, both $\frac{1}{\beta}$ and $\frac{1+\alpha\beta-\beta}{\alpha}$ converge to 1. Therefore $s(\alpha, \beta)$ converges pointwise to s^*. Finally, $s(\alpha, \beta)$ does not uniformly converge to s^* because $\bar{x}(\alpha, \beta)$ does not uniformly converge to x^*.

Proof of Proposition 5

1. Denote the set of optimal price(s) offered by a type y buyer by $P(y)$. Any element in $P(y)$ maximizes the type y buyer's payoff:

$$\Pi(y) = \max_p \{(y-p)F(\hat{x}(p)) + [1 - F(\hat{x}(p))]\beta\Pi(y)\}.$$

The envelope condition is $\Pi'(y) = F(\widehat{x}(p(y)))$, where $p(y) \in P(y)$. From the buyer's optimal investment condition, $\Pi'(y) = e'(y)$. In addition, we know that function $\widehat{x}(p)$ strictly increases in p, and, following the same argument as in the proof of Lemma 3, the CDF F is continuous and has support $[x^*, x_0]$. Therefore, $P(y)$ is single-valued and is a function of y in the steady state.

The function P strictly increases in y for the following reason. If $supp(G)$ is degenerate, then we have nothing to prove. Otherwise, consider any $y_1, y_2 \in supp(G)$ with $y_1 > y_2$. Denote $P(y_i)$ by p_i for $i = 1, 2$. Then the following two inequalities must hold:

$$(y_1 - p_1)F(\hat{x}(p_1)) + [1 - F(\hat{x}(p_1))]\beta\Pi(y_1) > (y_1 - p_2)F(\hat{x}(p_2)) + [1 - F(\hat{x}(p_2))]\beta\Pi(y_1),$$
$$(y_2 - p_2)F(\hat{x}(p_2)) + [1 - F(\hat{x}(p_2))]\beta\Pi(y_2) > (y_2 - p_1)F(\hat{x}(p_1)) + [1 - F(\hat{x}(p_1))]\beta\Pi(y_2).$$

We add the two inequalities and re-arrange, and we have

$$(y_1 - y_2)[F(\hat{x}(p_1)) - F(\hat{x}(p_2))] > [F(\hat{x}(p_1)) - F(\hat{x}(p_2))][\beta\Pi(y_1) - \beta\Pi(y_2)].$$

By the assumption on e, $e(y_1) - e(y_2)$ is strictly smaller than $y_1 - y_2$. Combined with the buyer's indifference condition that $\Pi(y_1) - \Pi(y_2) = e(y_1) - e(y_2)$, we have $y_1 - y_2 > \Pi(y_1) - \Pi(y_2) > \beta[\Pi(y_1) - \Pi(y_2)]$. Given this, the above inequality implies that $F(\hat{x}(p_1)) > F(\hat{x}(p_2))$. This proves that P strictly increases in y.

Following the same argument as in the proof of Lemma 3, the function H is continuous and has support $[r_S(x^*), r_S(x_0)]$. Then the monotonicity of P implies that the CDF G and G_e has support

$[\underline{y}, \bar{y}]$ and $G(\underline{y}) = G_e(\underline{y}) = 0$. To obtain the upper bound $\bar{y}$, note that $e'(\bar{y}) = \Pi'(\bar{y}) = 1$. The first equality follows from the indifference condition, and the second equality follows from the envelope condition and the fact that $P(\overline{y}) = x_0$. This means that $\bar{y} = y^*$ and $\pi = \Pi(\bar{y}) - e(\bar{y}) = y^* - x_0 - e(y^*)$. Similarly, to obtain the lower bound $\underline{y}$, note that $P(\underline{y}) = r_S(x^*)$ and

$$e'(\underline{y}) = \Pi'(\underline{y}) = \frac{F(x^*)}{1 - \beta(1 - F(x^*))}.$$

Calculate π based on the type $\underline{y}$ buyer's payoff,

$$\pi = \Pi(\underline{y}) - e(\underline{y}) = (\underline{y} - x^* - \beta c(x^*)) \frac{F(x^*)}{1 - \beta(1 - F(x^*))} - e(\underline{y}).$$

Plug in $\pi = y^* - x_0 - e(y^*)$ and $\frac{F(x^*)}{1-\beta(1-F(x^*))} = e'(\underline{y})$, and the condition becomes

$$y^* - x_0 - e(y^*) = (\underline{y} - x^* - \beta c(x^*))e'(\underline{y}) - e(\underline{y}). \tag{17}$$

The right-hand side strictly increases in $\underline{y}$. It equals zero at $\underline{y} = y_0$, which is strictly smaller than the left-hand side. It is strictly larger than the left-hand side at $\underline{y} = y^*$. Therefore, (17) uniquely determines $\underline{y}$, and $\underline{y}$ is strictly lower than y^*.

Next, we establish the existence and uniqueness of the equilibrium. The equilibrium price $P(y)$ is uniquely solved from the indifference condition $\Pi(y) - e(y) = \pi$ as

$$\begin{aligned} (y - P(y))e'(y) - e(y) &= y^* - x_0 - e(y^*) \\ \Rightarrow P(y) &= y - \frac{e(y) + y^* - x_0 - e(y^*)}{e'(y)}. \end{aligned}$$

In addition, combining the envelope condition of $\Pi(y)$ and the indifference condition that $\Pi'(y) = e'(y)$ for any y on the support, we can solve

$$F(\hat{x}(P(y))) = \frac{(1 - \beta)e'(y)}{1 - \beta e'(y)}.$$

Because the price function P is continuous and strictly increasing in y, the CDF F is solved uniquely. We can also plug the function P into H as calculated in (7) to solve G as

$$G(y) = \begin{cases} 0, \text{ if } y \in (-\infty, \underline{y}) \\ \frac{1+c'(\hat{x}(P(y)))}{1+\beta c'(\hat{x}(P(y)))}, \text{ if } y \in [\underline{y}, y^*], \\ 1, \text{ if } y \in (y^*, +\infty). \end{cases} \tag{18}$$

Finally, from the steady state condition, the CDF G_e is uniquely determined.

2. When we plug $\beta = 1$ and $\underline{y} = y^*$ into the right-hand side of (17), it becomes $y^* - x^* - c(x^*) - e(y^*)$ and it is strictly larger than the left-hand side. This implies that $\lim_{\beta\to 1} \underline{y} < y^*$. For any $y > \underline{y}$, $\hat{x}(P(y)) > x^*$. According to (18), $G(y) \to 1$ for any $y > \underline{y}$ when $\beta \to 1$. This also implies that for any $y > \underline{y}$, there exist $\breve{y} \in (\underline{y}, y)$ and $\breve{\breve{y}} \in (\underline{y}, y^*)$, such that

$$G_e(y) = \frac{\int_{\underline{y}}^{y} F(\hat{x}(P(\tilde{y})))dG(\tilde{y})}{\int_{\underline{y}}^{y^*} F(\hat{x}(P(\tilde{y})))dG(\tilde{y})} = \frac{F(\hat{x}(P(\breve{y})))G(y)}{F(\hat{x}(P(\breve{\breve{y}})))} = \frac{e'(\breve{y})(1-\beta e'(\breve{\breve{y}}))}{e'(\breve{\breve{y}})(1-\beta e'(\breve{y}))}G(y)$$

with both $\breve{y}$ and $\breve{\breve{y}}$ approaching $\underline{y}$ when $\beta \to 1$. Therefore, $G_e(y) \to 1$ when $\beta \to 1$ for any $y > \underline{y}$.

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